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A PROGRESSIVE INCOME-TAX.

BY PROFESSOR H. S. CARSLAW.

In this paper I propose to describe a scheme of progressive taxation which has been introduced in Australia in the "Income-Tax Act, 1915" (Commonwealth of Australia). As a novel illustration of the use of the Integral Calculus it may be of interest to readers of the *Mathematical Gazette.* Incidentally I desire to point out the need for care in using the term "rate of tax" when such a progressive tax is described. In the schedules of this Act the words are used in at least two different senses. And if the curves of the second degree, or any degree, are to be referred to in Acts of Parliament—a step the wisdom of which certainly may be questioned so long as the mathematical knowledge of the average man remains what it is—the nature of the curves might with advantage be more clearly stated than is done in this Act.

To take the simplest possible case of a progressive tax, let us suppose the tax arranged as follows:

On the 1st pound, the tax is $(a+\frac{1}{2}b)$ pence;

On the 2nd pound, the tax is $(a+\frac{3}{2}b)$ pence;

....

On the nth pound, the tax is $\left(a+\dfrac{2n-1}{2}b\right)$ pence.

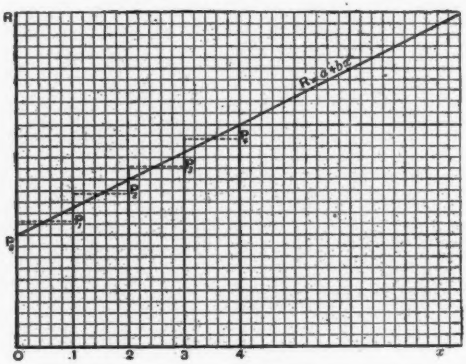
Then on £n the tax will be got by summing this Arithmetical Progression, and we have for the result $n(a+\frac{1}{2}nb)$ pence. We might say that the rate per £ on an income of £n is $(a+\frac{1}{2}nb)$ pence. However, this way of describing the tax would be unfortunate. It does not distinguish between the *rate at the nth pound* and the *average rate on £n.*

The change in the rate of the tax in this example has taken place at each increment of £1. It might have taken place continuously, just as Compound Interest can be reckoned from instant to instant, instead of yearly, half-yearly, or quarterly. Modifying the example, we shall assume that the rate in pence at the xth pound is given by the equation

$$R=a+bx.$$

Then the tax on £x would be $\int_0^x (a+bx)dx$ pence, or $x(a+\frac{1}{2}bx)$ pence.

It is interesting to examine these two cases graphically. Taking two rectangular axes, let the x-axis refer to sums in pounds sterling and the R-axis to rates in pence.



In the figure, the line $$R=a+bx$$ is drawn, and the ordinates at the points on the axis of x, whose abscissae are 1, 2, 3, etc., meet this line at P_1, P_2, P_3, etc.

The tax on the nth pound in the second example is given by the area of the trapezium whose top corners are P_{n-1} and P_n.

The dotted lines in the figure, through the middle points of P_0P_1, P_1P_2, etc., are at distances $a+\frac{1}{2}b$, $a+\frac{3}{2}b$, etc., from the axis of x. And the areas of the successive rectangles which we thus introduce into this figure give the tax on the successive pounds in the first example.

It will be seen that the tax on £n in these two cases is the same. And it should be noted that though the rate of the tax at the nth pound is $(a+bn)$, the tax on the nth pound is $\left(a+\dfrac{2n-1}{2}b\right)$. The distinction here referred to corresponds to that which meets the beginner when he is dealing with velocity in a straight line under constant acceleration. With the usual notation, the velocity at the time t is given by $V+ft$, and the distance travelled in the interval $(t, t+1)$ is $V+f(t+\frac{1}{2})$.

When the rate of tax at the xth pound is given by the equation

$$R=a+bx, \quad \text{.......................................(1)}$$

we might quite naturally say that the rate increases continuously in a curve of the first degree. And it is but a step to extend this to the cases

$$R=a_0+a_1x+a_2x^2 \quad \text{.......................................(2)}$$

and $$R=a_0+a_1x+a_2x^2+a_3x^3. \quad \text{..............................(3)}$$

These are the curves of the second and third degrees introduced into the Income Tax Act, as will be seen below.

In (2), the tax for such part of a man's income as lies between £x_1 and £x_2 would be

$$\int_{x_1}^{x_2}(a_0+a_1x+a_2x^2)\,dx,$$

and in (3) it would be

$$\int_{x_1}^{x_2}(a_0+a_1x+a_2x^2+a_3x^3)\,dx.$$

Let us now turn to the Schedules of the Act. The First Schedule deals with Income from Personal Exertion. It reads as follows:

FIRST SCHEDULE.

Rate of Tax upon Income derived from Personal Exertion.

For so much of the taxable income* as does not exceed £7,600 the rate of tax per pound sterling shall be Threepence and three eight-hundredths of one penny where the taxable value is One pound sterling, and shall increase uniformly with each increase of One pound sterling of the taxable income by three eight-hundredths of one penny.

For every pound sterling of taxable income in excess of £7,600 the rate of tax shall be sixty pence.

The rate of tax for so much of the taxable income as does not exceed £7,600 may be calculated from the following formula:

R = rate of tax in pence per pound sterling.

I = taxable income in pounds sterling.

$$R=\left(3+\frac{3}{800}I\right)\text{ pence.}$$

It will be noted that if we take the continuously progressive tax, the rate in pence at the xth pound in this case is given by the equation

$$R=3+\tfrac{3}{400}x,$$

and that, when $x=7{,}600$, we have $R=60$.

In this schedule it would have been more correct to say that

$$R=3+\tfrac{3}{400}I,$$

and that the tax on £I is $(3+\tfrac{3}{800}I)I$ pence.

The "rate of tax" referred to in the schedule as R is the "average rate."

The Second Schedule deals with Income from Property. It reads as follows:

SECOND SCHEDULE.

Rate of Tax upon Income derived from Property.

(*a*) For income of a taxable value not exceeding £546 the rate of tax shall be calculated from the following formula:

R = rate of tax in pence per pound sterling.

I = taxable income in pounds sterling.

$$R=\left(3+\frac{1}{181{\cdot}07}\right)\text{ pence.}$$

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*The words "taxable income" are used because of deductions allowed in certain circumstances.

(*b*) For income of a taxable value exceeding £546 but not exceeding £2,000 the rate of tax shall be calculated in the following manner:

The rate of the tax shall increase continuously with the increase of the taxable value of the income in a curve of the second degree in such a manner that the increment of tax per pound increase of taxable income shall be—

at a taxable income of	£546	11·713 pence
at a taxable income of	£600	12·768 pence
at a taxable income of	£700	14·672 pence
at a taxable income of	£800	16·512 pence
at a taxable income of	£900	18·288 pence
at a taxable income of	£1,000	20·000 pence
at a taxable income of	£1,500	27·600 pence
at a taxable income of	£2,000	33·600 pence

Note.—Amount of tax for taxable incomes of these amounts.				Equivalent to an average rate of—
£546 .	£13	13	8	6·015 pence
£600 .	16	8	9	6·576 pence
£700 .	22	8	2	7·597 pence
£800 .	28	13	2	8·597 pence
£900 .	35	18	2	9·576 pence
£1,000 .	48	17	9	10·633 pence
£1,500 .	93	15	0	15·000 pence
£2,000 .	157	15	7	18·933 pence

(*c*) For income of a taxable value exceeding £2,000 the rate of tax shall be calculated in the following manner:

For so much of taxable value as does not exceed £6,500, the rate of tax shall increase continuously with the increase of the taxable value of the income in a curve of the third degree in such a manner that the increment of tax per pound increase of taxable income shall be—

at a taxable income of	£2,000	33·600 pence
at a taxable income of	£2,500	40·000 pence
at a taxable income of	£3,000	45·300 pence
at a taxable income of	£3,500	49·600 pence
at a taxable income of	£4,000	53·000 pence
at a taxable income of	£4,500	55·600 pence
at a taxable income of	£5,000	57·500 pence
at a taxable income of	£5,500	58·800 pence
at a taxable income of	£6,000	59·600 pence
at a taxable income of	£6,500	60·000 pence

Note.—Amount of tax for taxable incomes of these amounts.				Equivalent to an average rate of—
£2,000 .	£157	15	7	18·9333 pence
£2,500 .	234	12	11	22·5258 pence
£3,000 .	323	13	7	25·8944 pence
£3,500 .	422	14	0	28·9851 pence
£4,000 .	529	14	5	31·7833 pence
£4,500 .	642	19	7	34·2921 pence
£5,000 .	760	18	1	36·5233 pence
£5,500 .	882	2	11	38·4936 pence
£6,000 .	1,005	11	1	40·2222 pence
£6,500 .	1,130	4	0	41·7806 pence

For every pound sterling of taxable income in excess of £6,500 the rate of tax shall be sixty pence.

This is a somewhat terrifying document: and one sympathises with the Treasurer in the task of explaining it to the House. Indeed, when Mr. Hughes, apologizing for its obscurity, was informed by the Leader of the Opposition that he quite understood the whole matter, he replied that if this were so his friend was much more fortunate than himself.

In section (*a*), dealing with incomes not exceeding £546, we have another case of the formula

$$R=a+bx,$$

giving for the tax on £x the sum of $x(a+\frac{1}{2}bx)$ pence.

It will be noticed that it would have been more correct to say that the rate is given by

$$R=3+\frac{2}{181{\cdot}07}I,$$

and the tax on £I is $\left(3+\frac{1}{181{\cdot}07}I\right)I$ pence.

Further, there is a discontinuity in the rate of tax in passing through £546 from (*a*) to (*b*).

In section (*b*) we meet the equation

$$R = a_0 + a_1 x + a_2 x^2.$$

It will be found by solving three linear equations that the "increments" for £1,000, £1,500 and £2,000 give the following values for a_0, a_1 and a_2:

$$a_0 = 0, \quad a_1 = \frac{23 \cdot 2}{10^3} \quad \text{and} \quad a_2 = -\frac{3 \cdot 2}{10^6}.$$

Also these values fit the "increments" for the other sums given in (*b*), except £546, where they give 11·7132288 instead of 11·713.

For the other incomes named the "increments" are exact: this one is correct to three places.

But it should be noted that when

$$R = a_0 + a_1 x + a_2 x^2,$$

the tax on the xth pound is $\int_{x-1}^{x} (a_0 + a_1 x + a_2 x^2) dx$, and in the wording of the schedule the "rate of tax" and the "increment" are confused. As a matter of fact "the increments of tax per pound increase of taxable income" are not as given in the Act.

In section (*c*), we meet the equation

$$R = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Taking the last four figures (£5,000 to £6,500) it will be found by solving four linear equations that the values for a_0, a_1, a_2 and a_3 are as follows:

$$a_0 = -5, \quad a_1 = \frac{25 \cdot 1\dot{6}}{10^3}, \quad a_2 = -\frac{3 \cdot 2}{10^6}, \quad a_3 = \frac{0 \cdot 1\dot{3}}{10^9}.$$

And these values fit the other "increments" exactly.

Though the wording of the schedule does not make this quite clear, the figures given in the Notes to (*b*) and (*c*) show that section (*a*) is to be used for the part of the income up to £546, (*b*) for the part from £546 to £2,000, and (*c*) for the part from £2,000 to £6,500.

And it would appear that the formula in (*a*), namely $R = 3 + \frac{1}{181 \cdot 07} I$, was chosen so that the average rate on £546 reckoned from sections (*a*) and (*b*) would be the same. However, a more exact calculation leads to

$$R = 3 + \frac{1}{181 \cdot 06} I,$$

instead of the form given.

To find the total amount of tax for an income between £546 and £2,000, we have then only to integrate

$$\frac{23 \cdot 1\dot{6}}{10^3} x - \frac{3 \cdot 2}{10^6} x^2.$$

e.g. for £1,000 we have $\int_0^{1000} \left(\frac{23 \cdot 1\dot{6}}{10^3} x - \frac{3 \cdot 2}{10^6} x^2 \right) dx$ pence, or £43. 17*s.* 9*d.*

For £2,000 we have $\int_0^{2000} \left(\frac{23 \cdot 1\dot{6}}{10^3} - \frac{3 \cdot 2}{10^6} x^2 \right) dx$ pence, or £157. 15*s.* 7*d.*

These agree with the figures in the Note to (*b*).

On the other hand, for an income of £3,000 we must take the part over £2,000 separately, using the formula of (*c*).

For this part we have

$$\int_{2000}^{3000}\left(-5+\frac{25\cdot1\dot{6}}{10^3}x-\frac{3\cdot2}{10^6}x^2+\frac{0\cdot1\dot{3}}{10^9}x^3\right)dx \text{ pence,}$$

which works out as £165. 18*s*. 0*d*.

Thus the amount of tax for £3,000 would be the sum obtained above for £2,000 plus £165. 18*s*. 0*d*.; in all, £323. 13*s*. 7*d*.

The other sums given in the Note to (*c*) will be obtained in the same way.

We have dealt so far with the original Act. Two months after it became law, it was followed by an Amending Act, in which it was enacted that

"Where the income of a taxpayer consists of income from personal exertion and income from property the rates of the income tax shall be:

(*a*) In respect of the income from personal exertion—the rate that would have been applicable if the total taxable income of the taxpayer had been derived exclusively from personal exertion; and

(*b*) in respect of the income from property—the rate that would have been applicable if the total taxable income of the ratepayer had been derived from property."

I leave the interpretation of this to the industrious reader. His task would have been an easier one if what the Treasurer called "the beautifully simple terms of the mathematician" had received kinder treatment in the drafting of the original Act.

Sydney, N.S.W., Dec. 9, 1915.

SO-CALLED CASES OF FAILURE IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

By Eric H. Neville.

(*Reprinted from the Journal of the Indian Mathematical Society, vol. vii.*, 1915.)

There are two ways in which the solution of a particular linear differential equation may "fail" although the solution of a more general equation obtained by replacing certain constants by parameters is complete. The two ways are distinct, but they may both be illustrated by the equation

$$(D-1)^2y=e^x, \qquad\qquad (1)$$

where D as usual stands for d/dx.

For the general equation

$$(D-l)(D-m)y=e^{nx}$$

the perfectly general solution is

$$y=\frac{1}{(n-l)(n-m)}e^{nx}+Ae^{lx}+Be^{mx},$$

A, B being independent arbitrary constants, but if we attempt to apply this solution to the particular equation (1), we find in the first place that the coincidence of n with l and m renders the first term infinite, and in the second place that the coincidence of m with l leaves us with only one effective constant, $A+B$. The method by which in the commoner text-books the passage from the general solution to that of a particular equation is made in such cases as this is unconvincing.

The fundamental problem is always the same: given a linear differential equation

$$G(y)=g(x, a_1, a_2, \ldots a_r) \qquad\qquad (2)$$

of any order, $a_1, a_2, \ldots a_r$ being constants, $G(y)$ involving x in any way and being homogeneous in y and a finite number of its derivatives, and $g(x, a_1, a_2, \ldots a_r)$ being any function of $x, a_1, a_2, \ldots a_r$, and given a function $F(x, p_1, p_2, \ldots p_r)$ of x and of r parameters $p_1, p_2, \ldots p_r$ which is not infinite irrespectively of x for the values $a_1, a_2, \ldots a_r$ of the parameters and is such that

$$G\{F(x, p_1, p_2, \ldots p_r)\} = h(p_1, p_2, \ldots p_r) f(x, p_1, p_2, \ldots p_r), \quad \text{.........(3)}$$

where the right-hand side is not identically zero for all values of $p_1, p_2, \ldots p_r$ and for the values $a_1, a_2, \ldots a_r$ of $p_1, p_2, \ldots p_r$ the function $f(x, p_1, p_2, \ldots p_r)$ takes the form $g(x, a_1, a_2, \ldots a_r)$, to construct as general a solution as possible of the equation (2).

Thus in the original example we know that

$$(D-1)^2 e^{px} = (p-1)^2 e^{px}, \quad \text{..............................(4)}$$

and we wish to utilise this knowledge in the formation of integrals of the particular equation (1).

It will be noticed that here the function $f(x, p)$, which is e^{px}, is actually the function $g(x, p)$; in practice this is usually the case, but it is quite irrelevant to the argument which follows, which enables us, for example, to deduce a solution of equation (1) from the more elaborate equation

$$(D-1)^2(xe^{px}) = 2(p-1)\{e^{px} + \tfrac{1}{2}(p-1)xe^{px}\}, \quad \text{.................(5)}$$

taking for $f(x, p)$ the expression $\{e^{px} + \frac{1}{2}(p-1)xe^{px}\}$.

We consider first the construction of a particular integral of equation (2), equation (3) being assumed.

If $h(a_1, a_2, \ldots a_r) \neq 0$, we have from (3) and the known relation between the functions $f(x, p_1, p_2, \ldots p_r)$, $g(x, a_1, a_2, \ldots a_r)$, and because the left-hand side is linear and homogeneous in y and its derivatives,

$$G\{F(x, a_1, a_2, \ldots a_r)/h(a_1, a_2, \ldots a_r)\} = g(x, a_1, a_2, \ldots a_r),$$

and therefore

$$y = F(x, a_1, a_2, \ldots a_r)/h(a_1, a_2, \ldots a_r)$$

is a solution of (2). In any case, p_k being any one of the parameters $p_1, p_2, \ldots p_r$, we have from (3) on account of the character of the operator G,

$$G\{\partial F/\partial p_k\} = (\partial h/\partial p_k) f(x, p_1, p_2, \ldots p_r) + h(p_1, p_2, \ldots p_r)(\partial f/\partial p_k), \quad (6)$$

and *precisely when* $h(a_1, a_2, \ldots a_r)$ *is zero* we can derive from (6) the r equations

$$G\{\partial F/\partial a_k\} = (\partial h/\partial a_k) g(x, a_1, a_2, \ldots a_r),$$

for $k = 1, 2, \ldots r$.

If then any of the functions $\partial h/\partial a_1, \partial h/\partial a_2, \ldots \partial h/\partial a_r$ are not zero, solutions of the original equation (2) are given by

$$y = \frac{\partial F}{\partial a_k} \Big/ \frac{\partial h}{\partial a_k}$$

for the corresponding values of k.

Thus, from equation (5) we have

$$(D-1)^2\{\partial(xe^{px})/\partial p\} = 2\{e^{px} + \tfrac{1}{2}(p-1)xe^{px}\}$$
$$+ 2(p-1)[\partial\{e^{px} + \tfrac{1}{2}(p-1)xe^{px}\}/\partial p], \quad \text{............(7)}$$

and from this it follows that

$$y = \tfrac{1}{2}[\partial(xe^{px})/\partial p]_{p=1} = \tfrac{1}{2}x^2 e^x$$

is a solution of equation (1), the right-hand side of equation (7) reducing, for $p=1$, to a constant multiple of e^x for the very reason that the factor $p-1$, which prevented our obtaining a solution of equation (1) by mere

substitution in equation (5), remains a factor of that part of the expression which is not itself a constant multiple, for $p=1$, of e^x.

But it may happen that all the functions of the form $\partial h/\partial a_k$ are zero. For example, if we are using the simple equation (4) for the purpose of solving equation (1), the function $h(p)$ of the only parameter p is $(p-1)^2$, and dh/dp as well as $h(p)$ is zero when p is equal to 1. And in fact differentiation of the equation (4) with respect to p gives us equation (5) itself, in the form

$$(D-1)^2\{\partial e^{px}/\partial p\}=2(p-1)e^{px}+(p-1)^2(\partial e^{px}/\partial p), \quad \text{.................(8)}$$

and the right-hand side of this equation does not reduce to a proper multiple of e^x for any value of p.

If, however, we differentiate the equation (6) with respect to any one of the parameters which it involves, we have for all values of the parameters and for all values of i, k from 1 to r,

$$G\left\{\frac{\partial^2 F}{\partial p_i\,\partial p_k}\right\}=\frac{\partial^2 h}{\partial p_i\,\partial p_k}f+\frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_k}+\frac{\partial h}{\partial p_k}\frac{\partial f}{\partial p_i}+h\frac{\partial^2 f}{\partial p_i\,\partial p_k},$$

and when h and its first derivatives with respect to the parameters p_i, p_k (which may coincide) vanish for the values a_1 a_2, ... a_r of p_1, p_2, ... p_r, we have

$$G\{\partial^2 F/\partial a_i\,\partial a_k\}=(\partial^2 h/\partial a_i\,\partial a_k)g(x, a_1, a_2, \dots a_r),$$

and if the second derivative $\partial^2 h/\partial p_i\,\partial p_k$ of h remains different from zero for the special values of the parameters, we can deduce a solution

$$y=\frac{\partial^2 F}{\partial a_i\,\partial a_k}\Big/\frac{\partial^2 h}{\partial a_i\partial a_k}$$

of our original equation. For example, from equation (8) we have

$$(D-1)^2\{\partial^2 e^{px}/\partial p^2\}=2e^{px}+4(p-1)(\partial e^{px}/\partial p)+(p-1)^2(\partial^2 e^{px}/\partial p^2),$$

an equation which yields us a solution of the equation (1) in the form

$$y=\tfrac{1}{2}(\partial^2 e^{px}/\partial p^2)_{p=1}=\tfrac{1}{2}x^2e^x$$

simply *because* the earlier equations (4), (8) were ineffective for this purpose.

The reasoning can be continued from step to step, and we conclude that if $h(p_1, p_2, \dots p_r)$ and all those of its partial derivatives of the first $m-1$ orders which are subsidiary to a particular derivative $\partial^m h/\partial p_1^{s_1}\,\partial p_2^{s_2}\dots\partial p_r^{s_r}$ that is, which are of the form $\partial^n h/\partial p_1^{t_1}\,\partial p_2^{t_2}\dots\partial p^{t_r}$, where $n<m$, $t_k \leqslant s_k$, vanish when p_1, p_2, ... p_r have the values a_1, a_2, ... a_r, but this derivative itself does not then vanish, a particular solution of equation (2) is given by

$$y=\frac{\partial^m F}{\partial a_1^{s_1}\,\partial a_2^{s_2}\dots\partial a_r^{s_r}}\Big/\frac{\partial^m h}{\partial a_1^{s_1}\,\partial a_2^{s_2}\dots\partial a_r^{s_r}}.$$

The common case is that in which there is only one parameter p, and only one constant a in the original equation, and the functions $f(x, p)$, $g(x, p)$ are the same; h is a function of p alone, F is a function of x and p, and the result takes the form that if h, dh/dp, d^2h/dp^2, ... $d^{m-1}h/dp^{m-1}$ all vanish for $p=a$, but $d^m h/dp^m$ does not then vanish, a solution of the equation

$$G(y)=g(x, a)$$

is given by

$$y=\frac{\partial^m F}{\partial a^m}\Big/\frac{d^m h}{da^m},$$

$F(x, p)$ and $h(p)$ being such functions that

$$G\{F(x, p)\}=h(p)g(x, p).$$

In this case we obtain only one solution of the equation, but in the general case we often obtain a number of solutions, which may or may not be independent.

To see how a multiplicity of parameters may be utilised, consider the equation

$$(D-1)^2(D-2)y=e^x+e^{2x}, \qquad \text{.................................(9)}$$

being given that

$$(D-1)^2(D-2)F(x, p, q)=h(p, q)\{e^{px}+e^{qx}\},$$

where

$$F(x, p, q)=(q-1)^2(q-2)e^{px}+(p-1)^2(p-2)e^{qx},$$
$$h(p, q)=(p-1)^2(p-2)(q-1)^2(q-2).$$

If we make both $p=1$ and $q=2$ in $h(p, q)$ or in any of its derivatives of the first three orders except $\partial^3h/\partial p^2\partial q$, the result is zero, but this one derivative has the value -2. Hence one solution of equation (9) is given by

$$y=-\tfrac{1}{2}\frac{\partial^3F(x, 1, 2)}{\partial p^2\partial q}=-\tfrac{1}{2}x^2e^x+xe^{2x}.$$

It is easy to see that in this example no other integral is derivable from the functions $F(x, p, q)$, $h(p, q)$.

For a second example, in which we cannot, without the use of complex numbers, resolve the problem into two simpler ones, as we should do in practice in the last example, we may solve the equation

$$(D^2-2D+2)y=e^x\sin x, \qquad \text{.................................(10)}$$

premising that

$$(D^2-2D+2)F(x, p, q)=h(p, q)e^{px}\sin qx, \qquad \text{.....................(11)}$$

where

$$F(x, p, q)=e^{px}[\{(p-1)^2+(1-q^2)\}\sin qx-2(p-1)q\cos qx], \qquad \text{........(12)}$$
$$h(p, q)=\{(p-1)^2+(1-q^2)\}^2+4(p-1)^2q^2. \qquad \text{............................(13)}$$

We see without difficulty that we may use either $\partial^2h/\partial p^2$ or $\partial^2h/\partial q^2$, and we find the two solutions

$$y=\tfrac{1}{4}e^x(\sin x-2x\cos x),$$
$$y=-\tfrac{1}{4}e^x(\sin x+2x\cos x),$$

each of which as a matter of fact includes a part usually absorbed into the complementary function.

Two comments are suggested by this last example. In the first place, we do not pretend to have here an exceptionally simple way of solving the equation (10); we are concerned only to shew that if the equation itself is contained under a more general form of which we have the solution, a solution of the equation is obtainable by a direct and unimpeachable process. And in the second place, instead of starting from a known result involving two parameters, we might have assumed that

$$(D^2-2D+2)F(x, p)=h(p)e^{px}\sin px, \qquad \text{......................(14)}$$

where

$$F(x, p)=e^{px}(\sin px+p\cos px), \qquad \text{..........................(15)}$$
$$h(p)=-2(p-1)(p^2+1), \qquad \text{.............................(16)}$$

and obtained the slightly different solution

$$y=-\tfrac{1}{4}e^x(\cos x+2x\cos x).$$

A similar possibility is clearly open in general. All that is required of $p_1, p_2, \dots p_r$ is that they should be capable of taking simultaneously the values $a_1, a_2, \dots a_r$, and subject to this condition they may be made from the commencement any convenient functions of a single parameter p. Thus

if $a_1, a_2, \ldots a_r$ are all different from zero we may put $p_k = a_k p$, and the particular equation to be studied will correspond to $p = 1$, or in any case we may put $p_k = a_k + p$ in the general equation which is given to us and consider the effect of the vanishing of p. But, since different equations may most naturally be discussed in different ways, there is evident advantage in limiting ourselves as little as possible in the general treatment.

(*To be continued.*)

MATHEMATICAL NOTES.

467. [X. 7.] My son, Mr. E. Mann Langley, has pointed out to me a simple geometrical construction which yields a remarkable connexion between the centimetre and the inch.

Its simplest form of statement is as follows:

If the side of an equilateral triangle is 5 inches long, then (with a relative error less than 1 in 5000) its altitude is 11 centimetres long.

It may be utilised thus:

To obtain a scale of centimetres from a scale of $\frac{1}{2}$ inches.

Take two points A and B 2·5 inches apart.

With centres A and B and radius 2·5 in. describe the semicircles *FBCD*, *EFAC*.

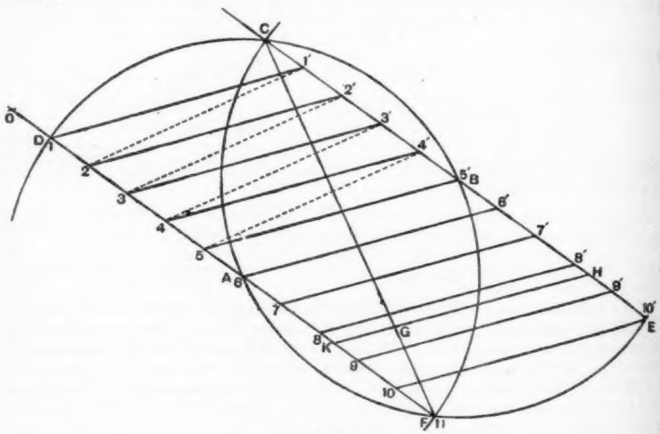


Fig. 1.

Graduate the diameters in $\frac{1}{2}$ inches as shown.

Then the lines 11′, 22′, 33′, etc. divide CF into 11 equal parts indistinguishable from 11 centimetres on any ordinary scale.

For $\quad CF = 2\frac{1}{2} \times \sqrt{3}$

$\qquad = 2\frac{1}{2} \times 1{\cdot}7320508$

$\qquad = 4{\cdot}330127$

10 cm. = 3·937079
1 cm. = ·393708
11 cm. = 4·330787
4·330127
·00066

Hence the relative error is less than ·00066/4·33, and equals ·00015 approx.

It should be noted that *without dividing CF as shewn*, we may obtain CG equal to any number of centimetres, say 8·4, by marking off half that number of inches (4·2). So that $CH=4·2$ in. $=OK$ and joining HK.

Millimetres are easily obtained, for

21′,	32′,	43′,	54′,	etc. cross CF at points
1,	2,	3,	4,	etc. millimetres below
11′,	22′,	33′,	44′,	etc.

Inspection of the figure leads to a second useful construction.

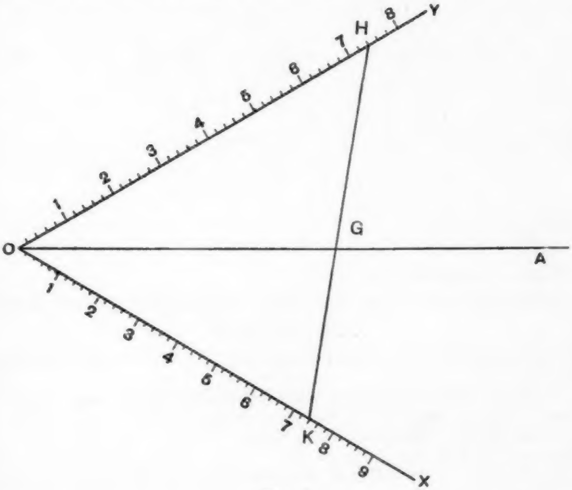


FIG. 2.

Let OX, OY be two straight lines at an angle of 60° and OA the straight line bisecting the angle XOY, along which it is required to mark off any required number of centimetres (say 7·4).

Along OY take $OH=3·7$ inches.

Along OX take $OK=\frac{5}{6}$ of 3·7 inches $=3·1$ inches approx.

Then if HK cuts OA in G,

$$OG=7·4 \text{ cm. approx.}$$

Both lines might be graduated; OY by marking 10ths, OX by marking off 12ths of inches. We have then only to join corresponding points.

E. M. LANGLEY.

468. [K[1]. 2. b.] *Given the base BC and the magnitude of the vertical angle A of a △ ABC.*

To find the locus of the centres of the escribed circle touching AC internally, i.e. of I_2.

$\angle BI_2C=\frac{1}{2}A$; ∴ the locus is a segment of a circle of which BC is the chord and an angle equal to $\frac{1}{2}A$ the contained angle. Its centre is on the circle BCA.

Now, consider the limiting positions of A:
(i) When A coincides with C,

BI_2 coincides with BC; $\therefore I_2$ falls at C.

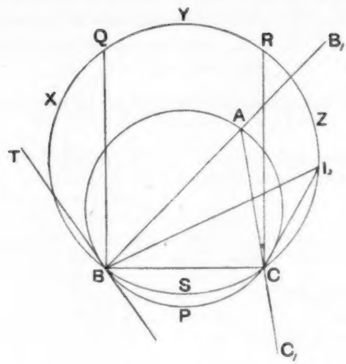


FIG. 1.

(ii) When A coincides with B,

CA coincides with CB and CI_2 falls along CR, the $\perp$ to BC at C;

$\therefore I_2$ falls at R.

[BA falls along BT, the tangent at B to the $\odot ABC$, and BR is the bisector of $\angle CBT$.]

We see, then, that as A moves anti-clockwise along its locus from C to B, I_2 moves along the arc CZR of the locus circle.

The locus circle is completed as follows:

1. The arc QXB is the locus of I_3.

2. The arc BSC is the locus of incentres of $\triangle$s in the supplementary segment BPC.

3. The arc RYQ is the locus of excentres of circles which touch the sides of $\triangle$s in the supplementary segment BPC, touching BC internally.

The continuity of these four broken parts of the locus circle is established by considering all positions of the circle which touches BC and the legs B_1A and C_1A of the external angle at A, as A moves round its locus.

Fig. 2. While A traverses the arc CAB, I_2 traverses the arc CZR of Fig. 1.

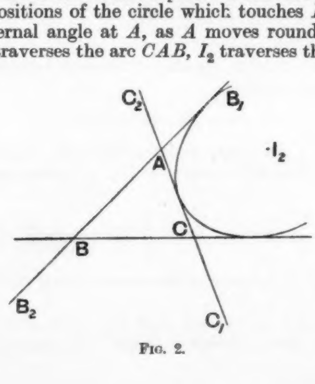


FIG. 2.

Fig. 3. After A has passed B it traverses the arc BPC, and I_2 traverses the arc RYQ of Fig. 1.

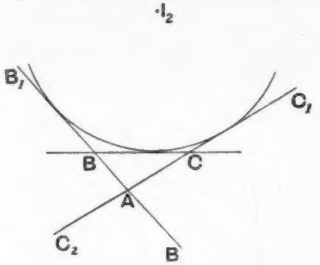


FIG. 3.

Fig. 4. After A has passed C it again traverses the arc CAB, and I_2 traverses the arc QXB of Fig. 1.

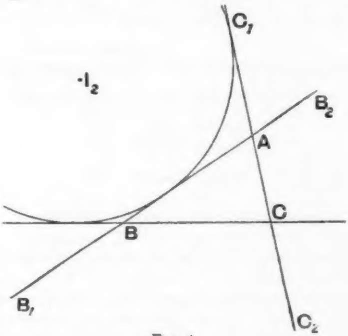


FIG. 4.

Fig. 5. After A has again passed B_1 it again traverses the arc BPC, and I_2, now an incentre, traverses the arc BSC of Fig. 1.

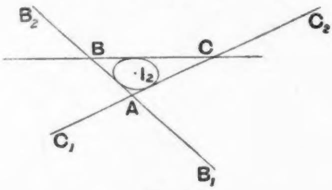


FIG. 5.

It is seen, therefore, that as A continuously traces out the circle CAB, I_2 continuously traces out the locus circle; and A must complete its circular path twice, while the locus circle is traced once; but it must be remembered that at the same time I_3 is also tracing out the locus circle, being always the other extremity of the diameter through I_2. F. C. BOON.

Dulwich Coll.

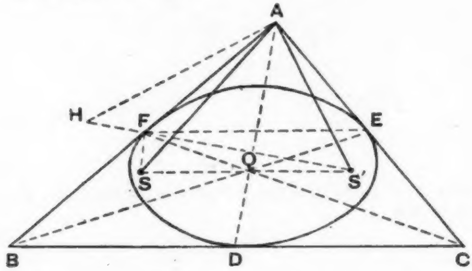
469. [K. 2. d.] (Vol. vi. p. 153, No. 349.) If D, E, F be the points of contact,

$$\triangle ABC - 2\triangle QBC = AFQE$$
$$= 2\triangle AHS'.$$

QA bisects EF; $\therefore$ $\triangle QAF = \triangle QAE$, and Q being the middle point of SS',

$$\triangle SAF + \triangle S'AF = 2\triangle QAF, \text{ etc.},$$

where H is the image of S in AB.



Therefore, if h, k be the altitudes of $\triangle$s ABC, QBC ($\angle BAC = \angle HAS'$),

$$1 - 2 \cdot \frac{k}{h} = \frac{AS \cdot AS'}{AB \cdot AC} = \frac{AS \cdot AS'}{2Rh};$$

or $$2R(h - 2k) = AS \cdot AS',$$

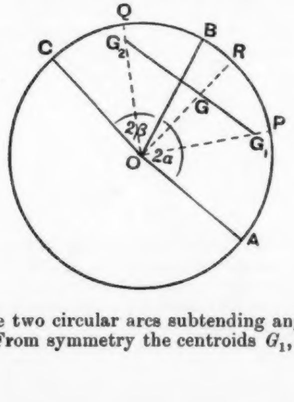
i.e. $$4R \cdot QN = AS \cdot AS'.$$

M. Ray.

Metropolitan Institution, Calcutta.

470. [R. 2. b.] *Elementary Method of investigating the Centroid of a Uniform Circular Arc.*

The method is based upon the theorem that if a body is divided into two parts with centroids G_1 and G_2 and each part is concentrated into a particle of equal mass at its centroid, the centroid of the body is the centroid of the two particles at G_1 and G_2, and therefore divides G_1G_2 inversely as the masses.



Let AB and BC be two circular arcs subtending angles $2a$ and 2β at the common centre O. From symmetry the centroids G_1, G_2 and G of AB, BC

and AC lie on the bisectors OP, OQ, OR of the angles which they subtend at the centre. Also, by the theorem quoted above, G_1, G, G_2 are collinear and

$$\frac{G_1G}{GG_2}=\frac{\beta}{\alpha}, \quad \text{.......................................(1)}$$

since the masses of the arcs are proportional to the angles at the centre.

Now $$\angle AOR=\angle ROC=\alpha+\beta;$$

therefore $$\angle POR=\angle AOR-\angle AOP=\beta$$

and $$\angle ROQ=\angle ROC-\angle QOC=\alpha.$$

Hence $$\frac{G_1G}{GG_2}=\frac{OG_1}{OG_2}\cdot\frac{\sin G_1OG}{\sin GOG_2}=\frac{OG_1\sin\beta}{OG_2\sin\alpha}. \quad \text{.........................(2)}$$

Equating (1) and (2), we have

$$OG_1 : OG_2=\frac{\sin\alpha}{\alpha} : \frac{\sin\beta}{\beta}.$$

Hence the ratio of $OG_1 : \dfrac{\sin\alpha}{\alpha}$ is independent of α, and therefore

$$OG_1=k\frac{\sin\alpha}{\alpha}=k\frac{\text{chord}}{\text{arc}},$$

the angle α being in circular measure.

To find k let the arc diminish and tend to zero; then the ratio

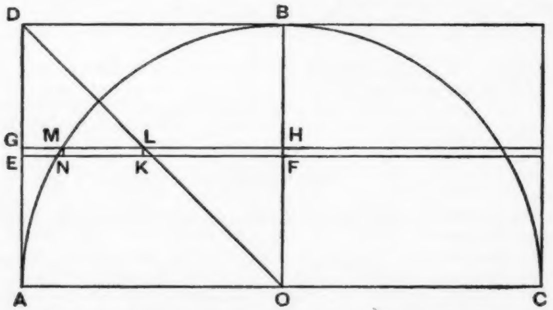
$$(\text{chord} : \text{arc})\rightarrow 1 \text{ and } OG_1\rightarrow\text{the radius};$$

hence finally, $$OG=\text{radius}\times\frac{\text{chord}}{\text{arc}}.$$

D. M. Y. SOMMERVILLE.

471. [K[1]. 16. g.] *To find the Volume of a Sphere.*

In the accompanying figure let the quadrant OAB, the square $OADB$, and the triangle ODB, generate respectively a hemisphere, a cylinder, and a cone by rotation about OB. Let HG and FE represent planes, parallel to the base of the cylinder, cutting corresponding slices out of the three solids.



Volume of slice cut out of cylinder $=\pi . GH^2 . HF$.

" " cone $=\pi . LH^2 . HF$ (approx.).

" " hemisphere $=\pi . MH^2 . HF$ "

But $$LH^2+MH^2=OH^2+MH^2=OM^2=HG^2.$$

Hence

Volume of slice cut out of cylinder = volume of slice cut out of cone

+ " " hemisphere.

Hence, by the usual method of summation,

Volume of cylinder $OAGH$ = volume of cone OLH

+ volume of solid $OAMH$.

Now let $OH=h, \quad OA=R.$

Then $$\pi R^2 h - \frac{\pi}{3} \cdot h^2 \cdot h = \text{volume of solid } OAMH.$$

Hence $$\text{Volume of solid } OAMH = \frac{\pi}{3}(3R^2 - h^2) \cdot h.$$

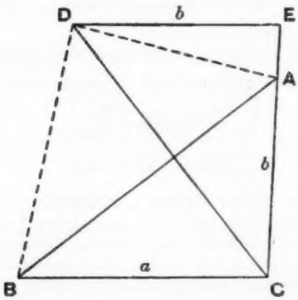
On putting $h=R$, we have

$$\text{Volume of hemisphere} = \frac{2\pi R^3}{3}.$$

J. O. Edwards.

472. [K¹. 1.] *Pythagoras' Theorem.*

ABC is a triangle, right-angled at *C*. Turn the triangle through a right angle round the centre of the square described on *BC* from the position *ABC*



into the position *DCE*. Then the quadrilateral *ADBC* consists of two triangles *DBC* and *DAC* having a common base of length *c* and of total height *c*.

The area of the quadrilateral is therefore $\frac{1}{2}c^2$. But the two triangles are respectively of areas $\frac{1}{2}a^2$ and $\frac{1}{2}b^2$.

$$\therefore \ c^2 = a^2 + b^2.$$

W. J. Dobbs.

REVIEWS.

Ueber die gleicheckig-gleichflächigen, discontinuerlichen und nicht-konvexen Polyeder. Von Prof. Dr. Max Brückner. (E. Karras, Halle.)

We have here a sequel to a previous work by the same author, *Vielecke und Vielfläche* (Teubner, Leipsic). The earlier work (to which we shall refer as *V. u. V.*), though published as long ago as 1900, seems little known in this country except to enthusiastic constructors of models such as Mr. W. Taylor and Prof. Crum Brown. Our own acquaintance with it dates from 1902, when Mr. Taylor, during the meeting of the British Association at Cambridge, exhibited a set of models, some of the most striking of which were reproductions of originals described and figured by Max Brückner in *V. u. V.*, a copy of which we were kindly allowed to inspect. It was a pleasure to get access

to the still larger collection shown by Prof. Crum Brown at the Edinburgh exhibition held in connexion with the Napier Tercentenary, and the pleasure was increased by the appearance, in the *Handbook* of the exhibition, of an interesting article upon it by the exhibitor.

Partly on account of the apparent scarcity of *V. u. V.* and partly on account of its bearing on the treatment of Polyhedra in the later work, we venture to give a sketch, necessarily brief and imperfect, as indeed is that of its sequel, of some of its important features.

Of its six sections A, B . . . F, A and B deal with polygons ; A giving the general theory, B dealing with special forms. The treatment is facilitated by considering pairs of figures which are polar reciprocals of one another.

We intend to use as abbreviations *g-e*, *g-f* and *g-k* as convenient abbreviations for *gleicheckig*, *gleichflächig* and *gleichkantig* respectively.

C, D, E, F are devoted to Polyhedra ; C giving the general theory ; D treating of that of Eulerian (*i.e.* convex) forms ; E discussing important special cases of these. F treats of non-convex polyhedra, both continuous and discontinuous.

In E one of the most interesting subsections is on the semi-regular solids called Archimedean and their polar reciprocals. Many of these will have been familiar to those fond of constructing models, especially as they or their analogous more generalised forms occur in crystals, but two others are less generally known, one a 38-faced figure derived either from the cube or the octahedron $(38 = 6 + 8 + 2 \times 12)$ and the other a 92-faced figure derived from either the dodecahedron or the icosahedron $(92 = 12 + 20 + 2 \times 30)$. The analytical investigation for the necessary construction leads in each case to a cubic equation which the author solves by Cardan's method. We follow the author in using A.V. for "Archimedean Variety" when referring to one of these.*

Among the historical remarks appended to this section, noticing some works of Badoureau, Pitsch, Hess and Feodorow, he quotes from the work of the second writer on semi-regular star polyhedra, a simple but very useful theorem which enables us to construct the *g-f* corresponding to the Archimedean variety of any *g-e*. If, as we hope, we are able to return in a separate note to the 38-faced solid just mentioned as having attracted the attention of Sharp, an interesting example may be given of its use.

Section F is devoted principally to the theory of stellar polyhedra as worked out by Cauchy, Jacoby, Bertrand, Cayley, Möbius, Wiener, Hess and others. A whole series of bodies has been added to the Kepler-Poinsot regular solids. As the figures even of these four are not often to be found in text-books, we give here reproductions of photographs of models constructed some years ago from the description in Rouché and Comberousse (Figs. 1, 2, 3, 4). It is very helpful to have these at hand when struggling with the more complicated ones figured in Dr. Brückner's works. In the interesting historical sketch of the various writers, it seems established that to Kepler is correctly attributed the discovery of two out of the four, although on the strength of some diagrams reproduced by Günther, it has been supposed that Jamitzer (1568) had anticipated him. The question as to whether Poinsot was aware of

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* Abraham Sharp, in his *Geometry Improved* (1717), draws attention to the first of these with the remark that "the Notion hereof was imparted by a Friend who understood so much of it as enabled him to draw the several Parts upon Paper or Past-board and fold them up into a due Form." At his friend's request he investigated the relations of its edges and faces, and found that its derivation from the cube depended on the cubic

$$x^3 + x^2 = \tfrac{1}{4}$$

obtaining for the solution $\quad x = \tfrac{1}{6}\sqrt[3]{19 + 3\sqrt{33}} + \tfrac{1}{6}\sqrt[3]{19 - 3\sqrt{33}} - \tfrac{1}{3}$

and working out various lines and areas with his well-known '*furor arithmeticus*' to 27 or more significant figures. His investigation concludes with an unexpected practical application. "Because this Body seems as advantageously composed for a Set of Dials as can well be contrived, I have calculated the Declination and $\left.\begin{matrix}\text{Re-}\\\text{In-}\end{matrix}\right\}$clination of all the Triangle Planes." Even while giving practical instructions for such a purpose his love of arithmetical nicety shows itself, Inclinations being given to ten thousandths of seconds (*e.g.* 9° 49′ 35″·0219). We hope to return in a future note to this solid, our own notion whereof was also first obtained "through a friend," by whom we were kindly presented with a set of mathematical stereograms dating from the early days of the stereoscope.

Kepler's discoveries is not discussed by the author, though he inclines to the opinion that Günther was right in answering it in the negative.*

Kepler's two star solids supply simple illustrations of two terms in constant use by the author in the later of his two works. Take, for instance, that shown in Fig. 1, and suppose it to be constructed by erecting on each face of a regular 12-hedron a pentagonal pyramid whose triangular faces have each base angle double the vertical one, the faces of the new star 12-hedron thus formed being regular star-pentagons. The planes of these pentagons enclose the original 12-hedron, which the author calls the 'Kern' of the new solid. Again, it is obvious that the vertices of these 12 pyramids are also vertices of a regular 20-hedron which thus encloses it. This 20-hedron he therefore calls the 'Hülle' of the star 12-hedron. The reciprocal figure shown in Fig. 2 has a regular 20-hedron for its 'Kern' and a regular 12-hedron for its 'Hülle.' In what follows we shall use the words 'Core' and 'Case' as convenient respective equivalents for these terms. A further illustration of their use is in the solids formed by 5 equal intersecting cubes and its reciprocal formed by 5 equal intersecting octahedra. The former of these has the regular p (Platonic) dodecahedron for its 'Case' and 30-faced rhombohedron for its 'Core,' the latter having consequently the (12+20)-faced A.V. figure for its 'Case' and the icosahedron for its 'Core.'

146 of the figures described in the text are illustrated by reproductions of photographs of actual models constructed by the author, many of them of great beauty. In giving an account of these he admits that not all the solids described in the text have illustrations given, partly because their construction was too complicated and partly because the chief requisite—*patience*—had not held out; we can easily believe it must have been sorely tried by many of those figured, which have complications enough. As the models had to be arranged according to size for photographic purposes, the order in the prints does not correspond to that of the descriptions, and some trouble is thus caused to the reader, who will do well to construct for himself a table of reference as he goes along, so that afterwards he can readily find the text corresponding to any model on the plates.

The later work seems to have been undertaken at the suggestion of E. Hess, who, in addition to the eight of the nine convex continuous solids made under special limitations, had already described some of the non-convex or discontinuous solids to be made when some of these restrictions are relaxed.

We intend to give (i) the author's classification of polyhedra; (ii) a slight sketch of his treatment with special reference to a few of the solids of which we are able to give photographic or diagrammatic reproductions.

A. I. CONVEX POLYHEDRA.

A. Continuous.

a. Regular.

(α) Of 1st kind:
The 5 Platonic Solids.

(β) Of higher kind:
The 4 Kepler-Poinsot Solids.

b. Not regular.

g-e and *g-f*, but not *g-k*.

(α) Of 1st kind:
Quadratic and Rhombic Sphenoids.

(β) Of higher kind; nine in number.

B. Discontinuous.

a. Regular.

'Core' and 'Case' regular.

(i) Group of 2 tetrahedrons.

(ii) Two symmetrical groups of 5 tetrahedrons.

(iii) Group of 10 tetrahedrons.

b. Not regular.

'Core' and 'Case' *g-e* and *g-f*, but not more regular.

(α) Unit bodies of 1st kind and regular.

(β) Unit bodies of 1st kind and not *g-k*.

(γ) Unit bodies of higher kind.

* It may be well to mention here that the memoirs on these 4 solids by Poinsot, Cauchy, Bertrand and Cayley have been republished in one volume (Nr. 151, M. 2.80) in Ostwald's *Klassike der exacten Wissenschaften* under the title *Abhandlungen über die Regelmässigen Sternkörper . . . übersetzt und herausgegeben von R. Hausner.*

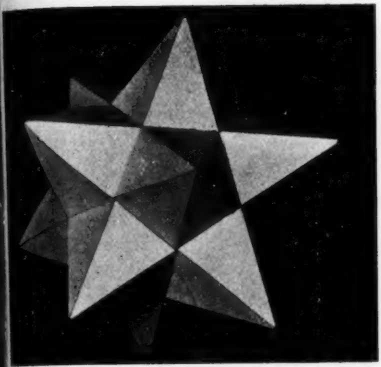

Fig. 1.

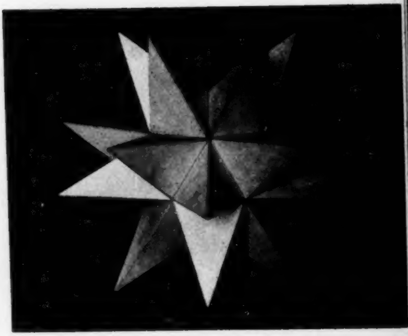

Fig. 2.

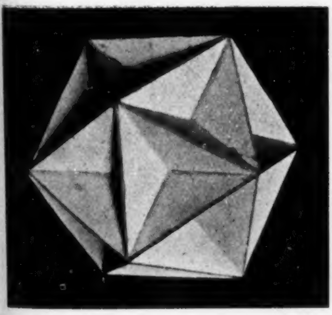

Fig. 3.

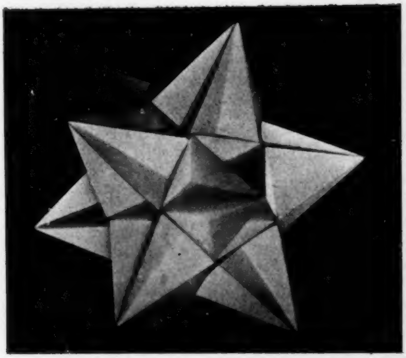

Fig. 4.

(*a*) Co
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(*a*) Co
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A. II. NOT CONVEX.

a. Of 1st class.

(*α*) Continuous.
3 of cubo-octahedral system.
5 of icosa-dodecahedral system.

(*β*) Discontinuous.
Combinations of (*α*).

b. Of 2nd class.

(*α*) Continuous.
Null-Polyhedra.
(i) Stephanoids of 1st and 2nd class and compound solids of which the unit bodies are generally stephanoids.

(*β*) Discontinuous.
Null-Polyhedra.
Unit bodies either stephanoids of the double pyramid type or continuous Null-polyhedra of the cubo-octahedral type.

B. MÖBIAN.

Six. All belonging to the icosa-dodecahedral system.

The existence of a few of the discontinuous solids given in the preceding list is so easily suggested by inspection of the Platonic solids as often to attract attention; the most obvious perhaps being that formed by the two equal tetrahedra obtained by 'hemigony' from the cube or 'hemihedry from the octahedron' (Kepler's *Stella octangula*) and the next in simplicity the two reciprocal figures mentioned above, as composed respectively of 5 cubes whose 40 vertices fall by twos on those of a regular dodecahedron and of 5 octahedra whose 40 faces fall by twos on those of an icosahedron.

The unit bodies whose groups form the others are simpler *ge-gf* bodies, generally sphenoids (square or rhombic) or stephanoids (also of two varieties). While the sphenoids are sufficiently familiar from their occurrence in the elements of Crystallography, the *stephanoids* seem to require a brief description.

These were invented by Hess and received their name from him, and are of two kinds. We describe a simple specimen of each.

(i) Let $ABCD\ldots$ be the base of a right n-sided prism, and $A'B'C'D'\ldots$ the opposite face, AA', BB', $CC'\ldots$ being lateral edges; then the crossed quadrilateral whose successive sides are $A'B$, BD', $D'C$, CA', is one of a set of $2n$ such quadrilaterals, and the solid formed by them is called a ***stephanoid of the first kind***. Its 'Case' is the prism from which it is derived, and its 'Core' a double pyramid of $2n$ faces.

(ii) Let $ABCDE\ldots$ be the base of a right $2n$-sided prism, and $A'B'C'D'E'\ldots$ the opposite face, and let a new solid be formed from it by cutting off the alternate corners A, B', C, $D'\ldots$ each by the plane through the three adjacent ones (B' by $A'BC'$, C by $BC'D$, and so on);* then the crossed quadrilateral whose successive sides $A'B$, BE', $E'D$, DA' is one of a set of $2n$ such quadrilaterals, and the solid formed by them is called a ***stephanoid of the second kind***.

The author, adopting the methods and technical terms of crystallography, treats separately of the solids derived:

(i) from the double pyramid system;
(ii) from the cubo-octahedral system;
(iii) from the dodeca-icosahedral system,

the analytical investigations being facilitated by the use of stereographic projections of the spherical nets corresponding to the complete figures of each of these systems. We hope to return to the net for (iii) on a future occasion.

The nature of the solids discussed is investigated by means of reference to axes. In (ii), for instance, A_1, A_2, $A_3\ldots$ denote the ends of the cubic axes; B_1, B_2, $B_3\ldots$ those of the rhombohedral, and C_1, C_2, $C_3\ldots$ those of the octahedral axes. In any solid A, B, C denote the lengths cut off along these axes by its faces. In comparisons C is supposed kept constant. The ratio of the A of the solid to the A of the cube is denoted by σ, that of the B of the solid to the B of the cube by τ; s and t being corresponding ratios for the reciprocal figure, so that $s\sigma = 1 = t\tau$. In (iii) analogous uses are made of G, C, B,

* A solid thus formed is called *kronrandig*. So also is its reciprocal.

corresponding respectively to the 12-dodecahedral, the 20-icosahedral and the 30-rhombohedral axes.

To each of the models constructed by the author and photographically represented (104 in number) corresponds a diagram showing one of the congruent set of faces by which it is formed, what parts of it are exposed to view in the completed model, and the lines in which it is cut by the other faces. An incidental value of these diagrams lies in the fact that they give worked-out solutions of interesting problems in Descriptive Geometry. In illustration of the method we have selected the 'continuous' solid whose 'Case' is the (6 + 8 + 12) A.V. polyhedron shown in isometric projection (Fig. 5).

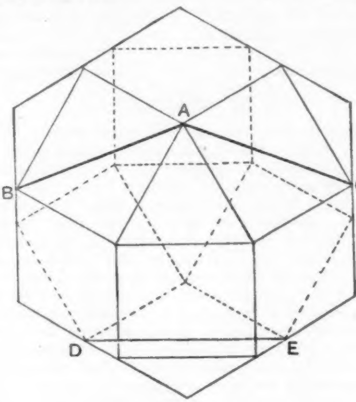

FIG. 5.

It is easy to see (i) that the diagonals of the three squares selected as shown by their projections AB, AC, DE, are in one plane;

(ii) that there are 24 such possible selections;

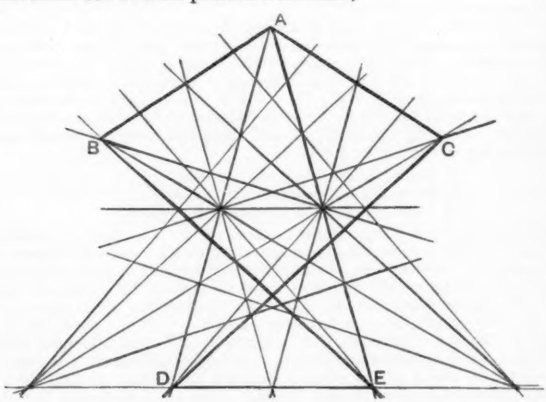

FIG. 6.

(iii) that these planes enclose a tetrakis-hexahedron (A.V.), which is therefore the 'Core' of the solid contained by the 24 faces obtained as $ABEDC$ is

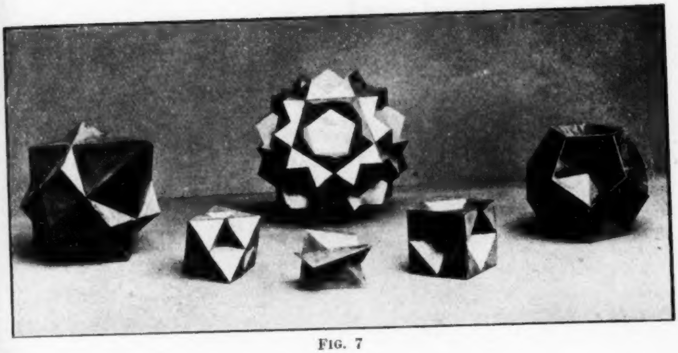

FIG. 7

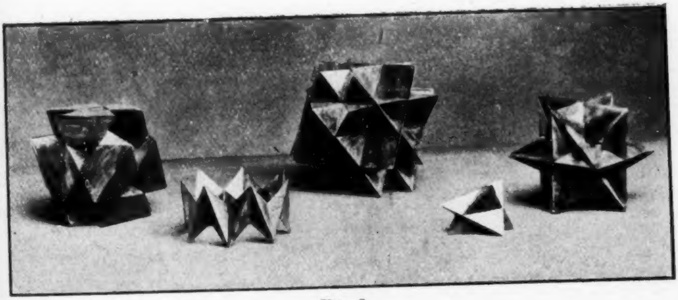

FIG. 8.

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above. Selecting one face of this solid, the lines are drawn on it, in which 22 of the other 23 faces cut it (Fig. 6). As an exercise in Descriptive Geometry we verified the author's diagram, obtaining its orthogonal projection on one of the faces of the cube, from which its 'Case,' the $(6+8+12)$ A.V. polyhedron, may be supposed to be cut, as V.P., an adjacent face being taken as H.P. It remains to describe briefly the solids shown in the other illustrations.

In Fig. 7 three of the six models represented are seen to be considerably larger than the other three. Of the three larger, the one on the left is composed of a pentagonal prism with equal edges and the double pyramid reciprocal to it, specially constructed to show the reciprocity of the two figures. The central one is formed by the intersections of the planes of two concentric regular dodecahedrons. That on the right is formed by the faces of a regular icosahedron produced until they meet in the edges of a regular dodecahedron—its 'Case' is in fact a regular dodecahedron, and its 'Core' a regular icosahedron. Of the three smaller solids, those on the right and left are specially designed to show the four regularly hexagonal sections of the cube and of the octahedron, and hence the six rhombohedral axes of these bodies. The central one is formed by the combination of two rhombic sphenoids.

In Fig. 8 the group consists of five solids, three large and two smaller ones. Of the three larger ones, that on the left is formed by producing the edges of a cube through its vertices until they meet the eight faces of a reciprocal octahedron; the sections by these eight faces forming star-hexagons. The central one is formed in a somewhat similar way. That on the right is a combination of six tetrahedra, each of which has for two opposite edges two of the median lines, at right angles to each other, of two opposite faces of a cube. The 'Case' is seen to be the cubo-octahedron; the 'Core' is the tetrakis-hexahedron, which has the Kelvin 14 face for its reciprocal.

Of the two smaller ones, the tiara-shaped one on the left hand is a 'stephanoid of the second rank'; its case and core being reciprocal 'kronrandige' solids having respectively $(2+2\times 7)$ faces and $(2+2\times 7)$ vertices.

The right-hand one is formed by the combination of two sphenoids. The models here shown are naturally among the simplest of those to be found pictured and described in the two works, though we are not without hope of making further progress in construction. EDWARD M. LANGLEY.

The Principles of Projective Geometry applied to the Straight Line and Conic. By J. L. S. HATTON. Pp. x+366. 10s. 6d. net. 1913. (Cam. Univ. Press.)

This is a thorough treatise on the point, line and conic, developed by means of the principles of projective and perspective geometry, the theory of anharmonic ratio being introduced at an early stage.

The book naturally divides itself into two parts, the first nine chapters being confined almost entirely to the properties of the point and line. The author commences by laying down the principle of duality, which is systematically used throughout, and then gives a very clear and simple explanation of the relation between projection and perspective, the foundations of the latter being laid in Euclid's porism that copolar triangles are coaxial. Homographic ranges are then dealt with, and it is shown that anharmonic ranges are projective, but although self-corresponding elements in the case of coaxial ranges are referred to and employed, their construction is deferred to a later period and obtained by projection on a circle or conic. This we think is a mistake, and might have been obviated by giving the elementary method due to Prof. A. Lodge (see *Math. Gazette* for April, 1909). The harmonic properties of a quadrilateral and quadrangle are then considered, and some very interesting examples worked out in full. Cap. VIII. is devoted to involution, which, being of supreme importance in the treatment of the conic, is dealt with very clearly and exhaustively. An interesting chapter on coplanar figures and problems of the first degree completes the first part of the book.

In the second part of the work Mr. Hatton's method of dealing with the conics will best be gathered from his own words at the beginning of Cap. X.:

"Art. 72. A circle is usually defined as a curve such that the distance of every point on it from a fixed point is constant. It immediately follows

(a proof being given in Art. 73) that it is the locus of the points of intersection of corresponding rays of two directly equal pencils. In Art. 92 a conic is defined as the locus of the points of intersection of pairs of corresponding rays of two projective pencils. Hence a circle is a particular case of a conic. Also since equal pencils are projected into projective pencils, the projection of a circle is a conic, and since the process of projection is reversible, a conic may be projected into a circle."

One cannot help feeling a certain amount of hesitation about accepting the above method of procedure. Are we justified in asserting, without proof, that the curve which is the locus of the points of intersection of pairs of corresponding rays of two projective pencils is a conic, meaning, of course, the section of a circular cone by a plane ? Chasles, in his *Traité de Géométrie Supérieure*, was of opinion that it required a proof, which he gave in Arts. 541-547, and in this he was followed by Henrici in his article on "Geometry" in the *Ency. Brit.* §§ 45-56, both writers showing that the curve locus possesses certain properties which entitle us to call it a conic. A more direct and quite elementary proof due to M. J. Delbalat was afterwards given by Chasles in Art. 8 of his *Traité des Sections Coniques*, and reproduced in Cremona's *Élements de Géométrie Projective*, Art. 114. Our author then proceeds to apply the principles of homography to the circle, proving the properties of the inscribed quadrangle and circumscribed quadrilateral, and then gives us two chapters dealing with special properties and systems of circles, concluding with proofs of the theorems of Carnot, Pascal and Desargues for the circle. Seeing that it is constantly recognised that the circle is a particular case of the conic, and that proofs of the above properties and theorems are afterwards fully given for conics, it would appear that a good deal of the work in these Chapters X., XI., XII. is redundant, and might have been omitted, references to the relation between the circle and conic being all that is required. We should then have been referred to proofs of the conic for properties of the circle instead of *vice versâ*, as on p. 172. Having cleared the circle out of the way, the author in Cap. XIII. enters on the study of the conic, which he carries out in the most thorough manner, distinguishing between the different species, and obtaining from the anharmonic property the various elements and focal properties of the curve. In Caps. XIV., XV., we have a full treatment of the theorems of Carnot, Pascal and Desargues, with proofs by the anharmonic theorem, and alternate proofs showing their inter-dependence, followed by a considerable number of very interesting deductions from each theorem.

In Cap. XVII. we meet with the construction of self-corresponding elements of two superposed projective forms, which is made to depend upon projection on a circle. With regard to this construction for two coaxial ranges, which our author terms the method of false positions, and which is the only one given in most text-books, we must remark that although theoretically perfect, it will in general be found quite unworkable in practice. The alternative method suggested by Chasles in his *Traité de Géométrie Supérieure*, Art. 263, is hardly any better, as it depends on the construction of five circles ; in fact, the only satisfactory method will be found to be that due to Prof. A. Lodge, referred to above. An easy test of this criticism is to apply it to Ex. (*a*) on p. 238, one of the simplest cases of this class of problem that could be given.

Cap. XVIII. contains a number of very interesting examples of loci and envelopes, pencils and ranges of conics, etc., treated by homographic methods, with solutions sufficiently full to enable the reader to follow them without difficulty. After dealing with conics through six points and touching six lines, we come to a chapter containing theorems concerning two conics which covers a very wide field, and gives *inter alia* the constructions of their common self-conjugate triangles, common chords both real and ideal, conics with double contact, and in Art. 133, a particularly neat direct proof of the harmonic locus, so often sought for in vain. With regard to the so-called alternative proofs of this, it would be better to call (*a*) the proof of the corresponding property for the circle, and (*b*) should be deleted, as of course it is quite without value as a proof. In the remaining chapters are to be found various *disjecta membra*, and the book concludes with non-projective proofs of a few properties of straight lines and circles.

Looking over the book after having made a study and companion of it for many months, one feels that it is an original work worthy to be put on a level with Chasles' *Traité des Sections Coniques*, with which it may well be compared, with the advantage in Mr. Hatton's favour that in the fifty years that have elapsed since Chasles' work appeared (1865), many new theorems and properties of conics have been discovered which are here given either in the text or amongst the examples, of which the author has given a very large number, with hints or solutions, always neat, and more or less full, as required. No one can read the book carefully without realising the power of the theory of anharmonic ratio when aided by the principles of projection and perspective. Of course there are classes of geometrical questions for the solution of which we must always be dependent on the methods of coordinate geometry, the processes of which are to a great extent mechanical, but in those to which the theory of cross-ratio is applicable the solution as a rule consists of a very few steps, the meaning of each of which is generally clear from the context. This also raises the question whether such a work as the one under review would not be better described as a work on cross-ratio. Seeing that by far the greater part of the results could be obtained by the method of cross-ratio alone, whilst few depend solely on projection, it would appear that the former is the predominant partner, who ought at least to appear as a member of the firm.

There is just one point in which the author might have made his work still more attractive, viz. by the insertion of references of a historical nature. Seeing that it is not a hundred years since the theory of cross-ratio was first employed, viz. by Möbius in 1827 and Chasles in 1829, and that the number of writers who have made use of it is small, and the historical sources fairly well known, it would not have been a difficult matter to have given a little of the information which we know from experience stimulates the interest of students in a subject. This would have saved him from the error made by so many writers on conics of attributing to Newton the theorem given on p. 290. It is, of course, Apollonius, Bk. III., Props. 16-23. The theorem referred to by Chasles under the name of Newton is the organic description of a conic given in the *Principia*, Bk. I. Sect. V. Lemma 21, and by some oversight, neither this, nor Maclaurin's theorem, from which Pascal's can be described, is given in the present work.

The book is got up in the best style of the Cambridge Press, and the figures are carefully drawn, the conics being really conics. Mr. Hatton has been fortunate in meeting with a field of research in which he had so few predecessors, and his clearness of statement and good readable style cannot fail to popularise both the methods employed and the subjects dealt with. We shall look forward with interest to the volume in which he promises to deal with the "imaginary" in connexion with the conic treated from a geometrical point of view.

JOHN J. MILNE.

A Budget of Paradoxes. By AUGUSTUS DE MORGAN. Second edition, edited by DAVID EUGENE SMITH. 2 vols. Pp. 402, 387. 30s. net. 1915. (Open Court Publishing Co.)

Of most mathematical school books of sixty or eighty years ago the world is well rid. "Where are the snows of yester year?" But De Morgan's publications on elementary mathematics sell to-day at second-hand book shops for more than their published price. He would have appreciated the weight of this evidence in his favour. As Dr. Rouse Ball indicates, in a biographical sketch which appeared in the *Gazette* (vol. viii. p. 42), De Morgan's great reputation now rests chiefly on the fact that he is one of the very few mathematicians of distinction who gave real thought and trouble to the difficulties and queer misconceptions of beginners and "lame dogs." Students of evolution must have often been impressed by the unexpectedness of the consequences of some development. It seems possible that the effect of the challenge problems which fill so large a space in the mathematical history of the seventeenth century, when Fermat, Brouncker, Wallis, the elder Bernoullis and Newton himself contended for personal and national honour, has been in the end injurious. For mathematicians were led to conceal the sequence of their thoughts, and the proof which Gauss, for instance,

gave to the world was that which would least help others to carry on his investigations.

De Morgan had a remarkable instinct for the natural development of a subject, and a gift, akin to that of a great general, for divining the stupid thing which his opponent, the pupil, was about to think or do. It is perhaps this side of his intellect which gives special charm to the work by which he is now best known outside the circle of professional teachers—the immortal *Budget of Paradoxes*; for he does not treat the circle squarers and the demolishers of Newton with mere contempt and scorn. Nothing was further from his mind than the bitterness worthy of theology which tramples on any suspicion of heterodoxy even in matters of exposition.

Unhappily the original edition of the *Budget of Paradoxes* has been for many years well nigh unprocurable, even by those who were willing to pay the two or three pounds demanded for a copy. Thus Professor Smith and the Open Court Company are entitled to sincere gratitude for adding to the valuable reprints already issued, a second edition, in two attractive volumes, of the *Budget*.

The temptation to "grangerise" by reprinting De Morgan's controversy with Hamilton from the *Athenaeum*, and by adding notes on subsequent heroes of unreason, has been resisted. Professor Smith has reprinted De Morgan's text, and added brief explanatory notes, giving bibliographic references, very short biographical notes of the persons mentioned, and translations of the numerous quotations occurring in the text. The amount of work and research these notes represent is immense.

It would be indiscreet as well as ungracious to take exception to the fact that in a book published by an American house and printed in America, De Morgan's spelling has been Americanised. The retort—Why did England show so little appreciation of a great book as to let it become unprocurable ?—would be too obvious.

One difficulty which oppresses a reviewer who wishes unreservedly to acknowledge instruction and information derived from the notes, and to express thankfulness to the editor for them, is solved by De Morgan himself. What should a reviewer do when he finds an obvious error in a valuable and worthy piece of work ? (See vol. 2, p. 162.) "Why should the faults of so good a writer be recorded ?" . . . Because "Nothing hurts a book of which few can understand the depth so much as a plain blunder or two on the surface."

The bibliographic notes constitute a marvellous, though modest and unassuming bit of research. Only those who have, once in a way at any rate, tried to hunt up a forgotten author, or to record the different editions of an old book, have any idea how hopeless is the task without painfully acquired skill, how often the wiliest hunter is baffled by the accidents of time, how many hours of work one name or one date in a note of a couple of lines may represent. And Professor Smith and his coadjutors have furnished hundreds of such notes.

But the translations require thorough revision. Too many instances will be found (*e.g.* vol. 1, p. 24, and vol. 2, p. 365), where the sense of a passage is missed.

That a second issue of this reprint may soon be called for will be the wish of all lovers of the memory of De Morgan. C. S. JACKSON.

Fundamental Conceptions of Modern Mathematics: Variables and Quantities, with a Discussion of the General Conception of Functional Relation. By ROBERT P. RICHARDSON and EDWARD H. LANDIS. Pp. xxii + 216. Price, cloth, $1.25 net, or 5s. net. 1916. (Chicago and London: The Open Court Publishing Company.)

This work is intended to examine critically the fundamental conceptions of mathematics, and the authors announce somewhat ambitiously that "it is, we believe, the first attempt made on any extensive scale" to make this examination, and that they mean to "clear away false doctrine preparatory to replacing it with true" (p. iii). The doctrines that the authors intend to clear away are those of the nominalistic school of mathematicians and also those of the logical school of Frege, Peano, and Russell. With respect to the nominalistic school, the task was not very difficult and had been done previously by Frege, but I think that the authors may claim an independent

success. In the closely allied notions that numbers are signs, that the "principle of permanence" is sufficient to extend our idea of number, and that a "variable" means what it is usually said to mean (pp. x, 2-3, 8-11, 97-110, 145-151, 153-161, 180), the authors seem to be correct. But they do not seem to have noticed the explanation of a variable given in Hardy's *Course of Pure Mathematics*, which would seem completely to avoid their just objections. Also, it is rather curious that the authors should not have seen the various publications in which Frege has exposed with irony and humour the various fallacies that noted mathematicians have committed about these very three points.

The authors also seem to be undoubtedly right in their remarks about inverse functions (pp. 173-177), and what is wrongly called "Dirichlet's definition of a function" (pp. 182-198). But on certain other points I cannot agree with them. The distinction that they try to draw between *equality* and *sameness* (pp. 5-6, 59-60) cannot, I think, be substantiated. The authors propose to restrict the word "same" to attributes perceived in the same object at different times. This would seem to make qualities like short-sightedness or colour-blindness come into questions which are purely logical. Further, what the authors call "two houses exactly alike" should obviously be not called, so they say, the "same" house but "like" or "equal" houses. Plainly the authors are thinking of houses which are alike except in position: if they were exactly alike they would be the same house.

Also, on p. 152 it is rather hasty to dismiss the logical claims of Frege on such a slight acquaintance with them, and it is not fair to deduce that Russell's definition of a cardinal number is absurd from the vague and inaccurate explanation given of the idea of *similarity* which is quite precise with Russell. The authors describe Russell as taking this idea "in a sense under which a thing may be said to be similar to itself." It cannot be said that such discussion as this "clears away" any errors there may be in Frege or Russell. It may be granted that in many places both of Russell's philosophical and of his popular writings there are ambiguities of expression which arise from impatience with pedantry. Such ambiguities, by the way, are far more rare with Frege. And the incompleteness of the process of clearing away is shown by the simple fact that the *Grundgesetze* of Frege and the *Principia Mathematica* of Whitehead and Russell, which may be described as the only books which attempt a strict proof of the logico-mathematical thesis, are not even mentioned.

Three other points may be referred to here. It seems to me to be correct to say that the number of fingers on my left hand is the *same* as that of those on my right hand, for the fingers of each hand are related to one and the same abstract number five. This is really the view of the authors (pp. 6-7), and is common sense: Frege and Russell simply go a step farther in the direction taken by common sense, and avoid postulating new entities by defining "the number five" as a definite one of what the authors call "value-classes." In the second place, it seems a pity to restrict the meaning of "function" to ordered sets (pp. 170-171, 198). Lastly, I heartily agree with the authors in what they say of De Morgan on pp. 120-121.

University of Edinburgh, Mathematical Department. Session 1915. Research Papers Nos. 1-11.

This series of research papers consists of papers published by Prof. Whittaker and his pupils in the *Proceedings of the Edinburgh Mathematical Society*, *Proceedings of the Royal Society of Edinburgh*, *Monthly Notices of the Royal Astronomical Society*, and *Proceedings of the London Mathematical Society*, bound in special blue covers with the above inscription. The papers of the series are in order: E. Lindsay Ince, "The Elliptic Cylinder Functions of the Second Kind"; E. T. Whittaker, "On a Class of Differential Equations whose Solutions satisfy Integral Equations"; Archibald Milne, "On the Roots of the Confluent Hypergeometric Functions"; Edward Blades, "On Spheroidal Harmonics"; E. T. Whittaker, "On an Integral Equation whose Solutions are the Functions of Lamé"; E. Lindsay Ince, "On a General Solution of Hill's Equation"; E. T. Whittaker, "On Lamé's Differential Equation and Ellipsoidal Harmonics"; E. T. Whittaker, "On the Functions which

are represented by the Expansions of the Interpolation Theory"; L. R. Ford, "On the Roots of a Derivative of a Rational Function"; A. G. Burgess, "Determinants connected with the Periodic Solutions of Mathieu's Equation"; L. R. Ford, "On the Oscillation Functions derived from a Discontinuous Function."

It is unnecessary here to do more than give the names of these "research papers"; all of them are well-known to mathematicians, and the only thing that remains for a reviewer to do is to congratulate Prof. Whittaker on these and other signs of the healthy activity of the Mathematical Department of Edinburgh University. PHILIP E. B. JOURDAIN.

First Year Mathematics for Secondary Schools. By E. R. BRESLICH. 4s. net. 1916. (Chicago University Press; published in England by the Cambridge University Press.)

This book is issued as one of the publications of Chicago University and its School of Education. The professors who, as editors, sign the first of the two prefaces, explain how the book came to be written. "The course of study in American high schools," they say, "is in process of extensive change. The change commenced with the introduction of new subjects. At first science began to compete with the older subjects; then came manual training, commercial and agricultural subjects, the fine arts, and a whole series of new literary courses." In the beginning the traditional subjects were not affected by this movement, and for a time the teachers of mathematics and classics were content to be left alone. This frame of mind was, however, but a temporary phase, and the editors say that it is now evident that the change in the high-school curriculum will not "come to an end until many changes have been made in the traditional subjects."

It must be allowed that hitherto there has been little real change in the teaching of mathematics, at any rate if all the schools of the country are taken into consideration. But in 1903 the School of Education of the University of Chicago began to seek for a solution of the problem of mathematical reorganisation, and they sought for a solution along the lines of fusion. In that year a tentative programme was drawn up for use in the University High School; it was revised the next year, and again in 1905; in 1906 it was printed. Other schools then began to use it. The book was revised again in 1909. In 1915 it was published in its present form, after having been re-written by Mr. Breslich, who is head of the department of mathematics in the High School.

The course is adapted for use by boys and girls aged about 13 or 14. It presupposes a knowledge of ordinary arithmetic and an ability to express one's meaning in good English. For the rest it begins at the very beginning with geometry and algebra, and leaves the pupil at the end with a knowledge of the main facts about parallels, the triangle, and the circle, and with a power to deal with problems which require quadratic equations for their solution. There are eighteen chapters, and for each of these there is provided at the end of the book a review and a set of supplementary questions.

In the second of the prefaces to the book, Mr. Breslich says that he has had the good fortune to get frank and helpful criticism from his colleagues and advice and criticism from various professors in the School of Education; the result of this is "that the book has become easily teachable." There is little doubt that this is a just claim. With slight alterations here and there the book could be used in English schools, and the Cambridge University Press might be willing to prepare an English edition of it if a number of English schools declared their willingness to give the book a trial. The Chicago University Press has already published a sequel in *Second Year Mathematics for Secondary Schools*. It must be remembered that, for the first year, the book does not claim rigour in definitions, axioms or principles, and that "insight has everywhere been the controlling consideration."

There are two other points to mention. The Chicago professors say that "the interest of to-day is in supervised study." Now study under supervision is not the same as study under compulsion, and there may be a difference in what we may call mental and moral atmosphere between a boys' school in England and a boys' school in America. The second point is one which

concerns the results of teaching. There is no doubt that in England long experience in teaching elementary mathematics tends to produce a sense of defeat in the teacher. The American professors are not unsympathetic in their reference to this failure ; the fact that they have encouraged this twelve-year experiment in Chicago is proof of that ; but they will not allow that the failure is necessitated by the circumstances of the case, and the words they use are worthy of quotation in full.

"It is useless to argue with a teacher who puts on the student body the blame when 25 per cent. of them are unable to profit by contact with himself. Such a teacher has no insight into the social relations of which he is a part ; he is absorbed in subject matter or in some other considerations remote from real life. He fails to realise the significant historical fact that the time has passed when the chief duty of the teacher is to eliminate students."

A First Course in Geometry. By CHARLES DAVISON, Sc.D. (Cambridge University Press.)

The sixty propositions contained in this little book would serve for a second year's revision, extension and consolidation of the geometry covered in the first year's course reviewed above. The order is not that of Euclid, but the book contains "the theorems and constructions in the first three books of Euclid's Elements which are of most consequence to beginners." Euclid's 16th proposition reappears as Proposition 4, and, in conjunction with Playfair's axiom, is used in the treatment of parallels ; of the seven chapters of the book two, the 3rd and the 6th, contain constructions only ; the other chapters are made up of theorems. Areas are dealt with in Chapters 4 and 7, and the first of these ends with the theorem of Pythagoras and its converse ; 112 sets of exercises are provided, never more than four exercises in any one set ; the book is compact and logical, and would be easily workable with a class of intelligent pupils who had already approached along other lines the facts with which it deals. T. M. A COOPER.

Annuaire pour l'an 1916, publié par le Bureau des Longitudes. Avec des notices scientifiques. Pp. 502+(A) 89+(B) 23+(C) 43. 50c. 1916. (Gauthier-Villars.)

It may be remembered that this hardy annual grew so bulky that it became necessary to divide the tables into two groups, one group to be taken in each year. This year it is the turn of the chemical and physical tables. Geography and statistics will follow in 1917. The astronomical tables are divided—deviation from the vertical in France, the value of g at various places, altitudes calculated by the barometer, stellar parallaxes, double stars, proper motions of stars, and stellar spectroscopy, will be dealt with next year ; then the complete set is found in the vols. 1915-1916 or 1916-1917. We seem to remember that the general accuracy of this little volume was called in question some years ago, but the names of G. Lippmann, G. Bigourdan, E. Picard, and H. Andoyer, attached to the Preface, should be a sufficient guarantee that, in spite of the war, everything has been done to make the data thoroughly reliable. M. Bigourdan's article on "La Pression moyenne et la régime des vents en France" is, as it were, a sequel to one some four years ago on mean temperatures, and is a valuable contribution to meteorological literature. This is the 121st year of issue of the *Annuaire*.

THE PILLORY.

Civil Service Commission [M.—612] Higher Mathematics. Paper I. No. 4.

Find the moment of inertia of a uniform circular disc about a line through a point in the circumference perpendicular to its plane.

Find also the times of a complete oscillation of the disc when suspended so as to swing about these two lines respectively if the diameter of the disc is equal to the length of the seconds pendulum (*i.e.* a pendulum which swings to and fro in 2 seconds). (Communicated by) W. C. F.

Obituary.

Harold Crabtree.

In Harold Crabtree Charterhouse lost a schoolmaster, and his colleagues a friend whom they could ill afford to lose.

A wrangler and a keen mathematician, he came to Charterhouse to teach Mathematics, but his deep piety leading him to claim a share in the teaching of Scripture, he was soon found to possess a remarkable knack in opening the eyes of boys to the beauties of Isaiah. Thus he gradually took up the teaching of English (his success with small boys in Shakespeare was remarkable) and, later on, of Classics in place of his mathematical work.

In his mathematical teaching he was very lucid and particularly neat in the arrangement of his own work, so that he excelled in inculcating good style; he always had his desk full of papers and solutions neatly docketed and immediately available when wanted, and he thoroughly understood how to work to a scheme.

At the time that he published his delightful book on *Spinning Tops and Gyroscopic Motion*—which has obtained an international reputation—only a small percentage of his school work was mathematical, and he was always somewhat apologetic for having "accidentally" written such a book.

His all-round capacity was shown in games as well as in work. As a boy (besides being Head of the School) he had been in the Football XI. and Captain of Cricket and of Fives. When he first returned as a master, he kept up his cricket, and played a good deal of fives: he was also a keen winter sportsman and a good skater, but owing to the heart and lung trouble, which in the end carried him off, he had to give up these things at an early age. The necessity of avoiding high altitudes was a severe blow. Of the school games I like to think that he most regretted fives. In this his tricky left-handed cut and strong volleying made him a redoubtable opponent and an admirable partner, and there was no more pleasant member of a weekly game.

I suppose what made him so universally popular was his cheeriness and his fund of genial and ever-ready repartee, and it was an object-lesson to see how thoroughly he remained his cheery self when he had to give up, one by one, the various recreations which he had most enjoyed.

All the last years of his life he was constantly troubled with shortness of breath if he exerted himself, for instance in walking uphill. But in spite of poor health, these years were probably the happiest in his life. In 1910 he married Miss Douglas, and two children came to them, a boy and a girl. Then in 1912 began new and interesting work as a housemaster, in which he succeeded admirably from the start in spite of the fact that he had to fill the place of the oldest and most popular of Carthusian housemasters.

Among his other activities may be mentioned: his keenness in the affairs of the Old Carthusian Club; his driving of a small motor-car; his writing and publishing a small volume of poems, the title of which was typical of the man, *Amicus Amicis*. He died early in 1915, and by a tragic coincidence his widow has since lost a brother, killed in action on the anniversary of her husband's death. C. O. T.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—The question discussed by Mr. Jackson and Professor Lodge on pp. 246-248 of your March number is of great importance in the teaching of Arithmetic and Algebra.

The issue involved is whether in the cases specified the use of brackets should be replaced by the adoption of certain rules.

These rules Mr. Jackson says are stated in many Arithmetical Books as follows:

1. Multiplications and divisions must be performed before additions and subtractions.

2. Multiplications and divisions must be performed in order (from left to right).

3. The word 'of' is, however, equivalent to a bracket.

Mr. Jackson approves Rule 1, but thinks that Rules 2 and 3 are artificial, unnecessary and indefensible.

It appears to me that Rule 1 cannot stand if Rule 2 is abandoned.

This will be seen at once from an example.

As Rule 1 contains the words "and divisions," it is legitimate to apply it to calculate the value of

$$9-6\div 3\times 2+4.$$

According to Rule 1 it is necessary to calculate first the value of $6\div 3\times 2$.

Now, if Rule 2 holds, this means $(6\div 3)\times 2=4$. But if Rule 2 does not hold, then it remains open to regard $6\div 3\times 2$ as meaning $6\div(3\times 2)=1$.

Hence, if Rule 2 is not binding, the expression to be calculated may mean either $9-4+4=9$, or $9-1+4=12$.

I desire to add two remarks on points in regard to which Mr. Jackson and Professor Lodge seem to misunderstand the attitude of those who, like myself, object to all three rules, on the ground that they increase unnecessarily the burden on the memory.

(*a*) Mr. Jackson implies that the objectors to the rules would regard such an expression as

$$ax+by+c$$

as ambiguous, and would say that it should be written

$$(ax)+(by)+c.$$

That is not my view. The apposition of algebraic quantities is in English books universally held to imply multiplication and to be equivalent to a bracket.

(*b*) Professor Lodge says that in some quarters there is apparently a desire to extend Rule 2 to a series of numbers connected by the four signs $+$, $-$, $\times$, $\div$; and he gives as an example the value of the expression

$$a\times b+c\times d+e\times f$$

when that is done.

I have never during upwards of thirty years of teaching written down such an expression without immediately making my meaning unmistakable by the insertion of brackets, though I am aware that many mathematicians do so.

In the hope that this correspondence may lead to a thorough discussion of the matter amongst teachers, I append to this letter a statement of the ambiguities which seem to me to arise when the above-mentioned rules are substituted for the use of brackets.—I remain, yours faithfully,

M. J. M. Hill.

University College, London,
2nd May, 1916.

On the possibilities of ambiguity which may arise from the omission of brackets from an algebraic expression.

1. If a number of terms are connected together by the *direct* sign of addition, there is no ambiguity in the meaning of the whole expression, *e.g.* $a+b+c$ means $(a+b)+c$, and it is also equal to $a+(b+c)$.

2. If a number of terms are connected together by the *direct* sign of multiplication there is no ambiguity in the meaning of the whole expression, *e.g.* $a \times b \times c$ means $(a \times b) \times c$, and it is also equal to $a \times (b \times c)$.

3. Consider next an expression containing both the *direct* sign of addition and its *inverse*.

If there is only one minus sign, and if further it comes before the last term there is no ambiguity, *e.g.* $a+b-c$ means $(a+b)-c$, and it has the same value as $a+(b-c)$.

If, however, the expression contains a minus sign which does not stand in front of the last term, a convention is required to remove ambiguity. The following convention is in universal use. It is that the symbols of operation are to be evaluated from left to right. It is in accordance with the practice in all European languages of writing from left to right, and so causes but little, if any, strain on the memory.

In accordance with it $\qquad a-b+c$

means $(a-b)+c$ and not $a-(b+c)$.

4. Consider next an expression containing both the *direct* sign of multiplication and its inverse.

If there is only one sign of division, and if further it comes before the last term, then there is no ambiguity; *e.g.*

$$a \times b \div c$$

means $(a \times b) \div c$, and it has the same value as $a \times (b \div c)$.

If, however, the expression contains a sign of division which does not stand in front of the last term, then a convention is required to remove ambiguity.

In some text-books it is stated quite explicitly that in an expression containing signs of multiplication and division only the symbols of operation are to be evaluated from left to right.

According to this convention, $a \div b \times c$ means $(a \div b) \times c$, and not $a \div (b \times c)$.

In some English text-books this convention is assumed without being explicitly stated.

But it is not universally adopted, as would appear from note 166 on page 44 of tome 1, volume 1, fascicule 1 of the French *Encyclopédie de Mathématiques*, which contains the following passage:

"L'écriture $a:bc$ prête à ambiguité. Contrairement à la convention de E. Schröder, on entendrait plutôt en France par $a:bc$ non pas $(a:b)c$ mais $a:(bc)$."

The difference in the notation employed makes an entirely reliable application of this statement somewhat difficult.

It is, I think, a problem in psychology to explain why the expression

$$a \div b \times c$$

is regarded as ambiguous by those, or at least by some of those, who regard $a-b+c$ as clear.

In my own practice I should avoid the use of $a \div b \times c$, writing it either $(a \div b) \times c$ or $a \div (b \times c)$ according to the meaning I wished to convey.

5. Let us now pass to an expression containing all the four signs of addition, subtraction, multiplication and division, but no brackets. To avoid misunderstanding, I desire to say that I would never use such an expression. I purpose only to set down here the substance of the con-

ventions which must be adopted by those who think it right to employ such expressions.

To see what these are, let me start from the algebraic expression

$$a-\frac{bc}{d}+e.$$

Now $\frac{bc}{d}$ has the same value as either of the expressions

$$[(b\times c)\div d] \quad \text{or} \quad [(b\div d)\times c].$$

Hence the original expression is equivalent to

$$a-[(b\times c)\div d]+e \quad \text{or} \quad a-[(b\div d)\times c]+e.$$

If now all the brackets be omitted, we are left with the expressions

$$a-b\times c\div d+e \quad \text{and} \quad a-b\div d\times c+e, \quad \ldots\ldots\ldots\ldots\ldots\text{(I.)}$$

as equivalents to

$$a-\frac{bc}{d}+e, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II.)}$$

and it is necessary to enquire what rules must be adopted to make it possible to return from either of the expressions (I.) to the expression (II.).

It seems to me that the rules are three in number.

(i) In an algebraic expression containing the signs of addition and subtraction only, but no brackets, the symbols of operation must be evaluated from left to right.

(ii) In an algebraic expression containing the signs of multiplication and division only, but no brackets, the symbols of operation must be evaluated from left to right.

(iii) In an algebraic expression containing all the four signs (or at least three of them) any plus or minus sign, together with the next following plus or minus sign, are to be regarded as constituting a bracket. The terms inside a bracket so constituted are to be evaluated by Rule (ii).

[For simplicity I leave out of the statement of this third rule the cases in which a bracket begins at the beginning of the whole expression, or a bracket ends at the end of the whole expression.]

This third rule is the equivalent of that which is usually stated, "Multiplications and divisions must be performed before additions and subtractions."

6. The main issue is whether it is necessary to require a knowledge of, and adhesion to, Rules (ii) and (iii). There is no question about Rule (i), which is universally accepted.

It is of course true that all three rules can be avoided by the use of brackets.

Rule (i) is universally accepted, and being in accordance with the order of writing in European languages involves no appreciable strain upon the memory.

Rule (ii) is also in accordance with that order; but in regard to it we are faced with the facts set out in paragraph 4 above. It is not universally adopted. To my mind it is far less objectionable than Rule (iii), which, in combination with Rule (ii), has some very curious consequences.

If Rule (iii) be applied to the expressions

$$a+b\times c \quad \text{and} \quad a-b\times c,$$

it results in the evaluation of the symbols of operation from right to left, a direct reversal of the order of writing in European languages, *and therefore confusing to beginners.*

If Rule (ii) be applied to the expression

$$a \div b \times c$$

it interprets it to mean $$(a \div b) \times c,$$

which is equal to $$a \div \left(\frac{b}{c}\right).$$

So that in accordance with Rules (ii) and (iii), whilst

$$a + b \times c \quad \text{means} \quad a + (bc),$$
$$a - b \times c \quad \text{means} \quad a - (bc),$$
$$a \div b \times c \quad \text{means} \quad a \div \left(\frac{b}{c}\right).$$

This seems to me to show that it is possible to draw a distinction between bc and $b \times c$.

The first is a single term, the multiplication of b by c *has been performed*; the second is the product of two terms, the explicit insertion of the sign of multiplication implies that the multiplication of b by c *has yet to be performed*. The multiplier c has not yet operated. If therefore the expression $b \times c$ is preceded by something else, the meaning is in doubt.

The only advantages that Rule (iii) possesses are that it saves some expense in printing and a little labour in writing, but in my opinion the saving obtained is dearly purchased by the ambiguity it leaves behind.

The strain involved on the memory in learning Mathematics is so great that it should be diminished by the removal of all unnecessary rules. To this class, in my opinion, Rule (iii) belongs.

Would it not be better that it should be abandoned? M. J. M. Hill

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